Homology of racks and quandles

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Slides for Lecture 3

6. Quillen homology

Relative abelianisation and Beck modules

$\mathbf{C}$ = a cat like **Groups**, **Racks**, **Quandles**, **Sets**, G–**Sets**,...

$\mathbf{Ab}(\mathbf{C})$ = abelian group objects = modules over a ring

X = an object in $\mathbf{C}$

$\mathbf{C}/X$ = cat of objects over $X = (Y \to X)$

$$\mathbf{C}/X \underset{\text{forget}}{\overset{\mathrm{Ab}_X}{\rightleftarrows}} \mathbf{Ab}(\mathbf{C}/X),$$

The objects in $\mathbf{Ab}(\mathbf{C}/X)$ are called **Beck modules** over X.

Example: groups

$$\mathbf{Ab}(\mathbf{Groups}) = \mathbf{Abel} = \mathbb{Z}\text{–}\mathbf{Mod}$$

$$\mathrm{Ab}(G) = G/[G,G] = \mathbb{Z}G/([g \cdot h] = [g] + [h])$$

$$\mathbf{Ab}(\mathbf{Groups}/G) = \mathbb{Z}G\text{–}\mathbf{Mod}$$

$$(A \to G) \longmapsto \mathrm{Ker}(A \to G)$$

$$(K \rtimes G \to G) \longleftarrow\!\!\mid K$$

$$\mathrm{Ab}_G(G) = \mathrm{I}G = \mathrm{Ker}(\mathbb{Z}G \to \mathbb{Z})$$

Example: quandles and knots

$$\textbf{Ab}(\textbf{Quandles}) = \mathbb{Z}[t^{\pm 1}]\text{–}\textbf{Mod}$$

Example. $\mathsf{Ab}(\mathsf{Q}_K)$ = (extended) Alexander module of knot K

Many non-trivial knots have trivial Alexander module...but:

Theorem (S). The **Alexander–Beck module**

$$\mathsf{Ab}_{\mathsf{Q}_K}(\mathsf{Q}_K)$$

detects the unknot.

It lives in an abelian category and does not involve homology!

The cotangent complex

For homology, derive the functor $Y \mapsto \mathrm{Ab}_X(Y)$ at $Y = X$:

Choose a free resolution $\mathrm{F}_\bullet \to X$ (so $\mathrm{F}_\bullet \in \mathbf{sC}/X$).

Then compute the (relative) abelianisation.This, or the associated chain complex, is the **cotangent complex**

$$\mathbb{L}_X(X) = \mathrm{Ab}_X(\mathrm{F}_\bullet) \in \mathbf{sAb}(\mathbf{C}/X).$$

Its homology $\mathrm{H}_\bullet \mathbb{L}_X(X)$ is the **Quillen homology** of X.

It lives in the abelian category $\mathbf{Ab}(\mathbf{C}/X)$ of Beck modules.

Example: the Quillen homology of free objects

If X is free, then the identity

$$F_\bullet = X$$

is a free resolution!

The space $F_\bullet = X$ is discrete, and so is the cotangent complex

$$\mathbb{L}_X(X) = \mathrm{Ab}_X(X).$$

Then $\mathrm{Ab}_X(X)$ is the Quillen homology in dimension 0.

The higher Quillen homology vanishes.

Free resolutions — existence

For all our categories $\mathbf{C}$, we have an adjunction

$$\mathbf{Sets} \underset{\text{forget}}{\overset{T = \text{free}}{\rightleftarrows}} \mathbf{C},$$

and this gives free resolutions:

$$\begin{array}{ccccc}
T(X) & \begin{smallmatrix}\longleftarrow \\ \longrightarrow \\ \longleftarrow\end{smallmatrix} & T^2(X) & \begin{smallmatrix}\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow\end{smallmatrix} & \cdots \\
\downarrow & & & & \\
X & & & &
\end{array}$$

Coefficients and Quillen cohomology

We would like to replace the abelian category

$$\mathbf{Ab}(\mathbf{C}/X)$$

of Beck modules over X by the abelian category

$$\mathbf{Abel}$$

of abelian groups. It is easier and independent of $\mathbf{C}$ and X.

Quillen **cohomology** with **coefficients** in a Beck module M

$$\mathrm{D}^\bullet(X; M) = \mathrm{H}^\bullet\mathrm{Hom}_{\mathbf{Ab}(\mathbf{C}/X)}(\mathbb{L}_X(X), M)$$

does just that!

From homology groups to a homology theory

Theorem (S). Rack cohomology and quandle cohomology with trivial coefficients agree with Quillen's cohomology for these categories, up to a shift.

Sketch of proof

First check this for **free** racks/quandles. The standard complex was computed by Farinati–Guccione–Guccione; the Quillen cohomology of free objects is trivial.

Then, reduce the general case to the free one using free resolutions, as before.

□

On the one spectral sequence — vertical first

Let $R \leftarrow F_\bullet$ be a free resolution and A an abelian group.

$$E^0_{p,q} = \mathrm{Hom}(\mathrm{CR}_p(F_q), A).$$

$\mathrm{CR}_p(R) \leftarrow \mathrm{CR}_p(F_\bullet)$ is a free resolution of a free abelian group.

$$E^1_{p,q} = \mathrm{Ext}^q(\mathrm{CR}_p(R), A) = \begin{cases} \mathrm{Hom}(\mathrm{CR}_p(R), A) & q = 0, \\ 0 & q \neq 0. \end{cases}$$

Then, the differential in the p–direction leaves

$$E^2_{p,0} = \mathrm{HR}^p(R; A)$$

in the row $q = 0$ and zeros elsewhere.

The bicomplex spectral sequence E^0

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\Big\downarrow \partial & \Big\downarrow \partial & \Big\downarrow \partial \\
\mathrm{Hom}(\mathrm{CR}_0(F_2), A) & \mathrm{Hom}(\mathrm{CR}_1(F_2), A) & \mathrm{Hom}(\mathrm{CR}_2(F_2), A) \\
\Big\downarrow \partial & \Big\downarrow \partial & \Big\downarrow \partial \\
\mathrm{Hom}(\mathrm{CR}_0(F_1), A) & \mathrm{Hom}(\mathrm{CR}_1(F_1), A) & \mathrm{Hom}(\mathrm{CR}_2(F_1), A) \\
\Big\downarrow \partial & \Big\downarrow \partial & \Big\downarrow \partial \\
\mathrm{Hom}(\mathrm{CR}_0(F_0), A) & \mathrm{Hom}(\mathrm{CR}_1(F_0), A) & \mathrm{Hom}(\mathrm{CR}_2(F_0), A)
\end{array}
$$

Compute the homology of the vertical differential to get E^1.

The bicomplex spectral sequence E^1

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\mathrm{Ext}^2(\mathrm{CR}_0(R), A) & \xrightarrow{\partial} & \mathrm{Ext}^2(\mathrm{CR}_1(R), A) & \xrightarrow{\partial} & \mathrm{Ext}^2(\mathrm{CR}_2(R), A) \\
\mathrm{Ext}^1(\mathrm{CR}_0(R), A) & \xrightarrow{\partial} & \mathrm{Ext}^1(\mathrm{CR}_1(R), A) & \xrightarrow{\partial} & \mathrm{Ext}^1(\mathrm{CR}_2(R), A) \\
\mathrm{Hom}(\mathrm{CR}_0(R), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_1(R), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_2(R), A)
\end{array}
$$

The higher Ext groups vanish, so we can simplify.

The bicomplex spectral sequence E^1

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
\mathrm{Hom}(\mathrm{CR}_0(R), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_1(R), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_2(R), A)
\end{array}
$$

Compute the homology of the horizontal differential for E^2.

The bicomplex spectral sequence E^2

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\mathrm{HR}^0(R;A) & \mathrm{HR}^1(R;A) & \mathrm{HR}^2(R;A)
\end{array}
$$

We are done!

On the other spectral sequence — horizontal first

$$E^0_{p,q} = \mathrm{Hom}(\mathrm{CR}_p(F_q), A)$$

First, the homology in the p–direction:

$$E^1_{p,q} = \mathrm{HR}^p(F_q; A)$$

This is non-zero only in the two columns $p = 0$ and $p = 1$.

The column $p = 0$ is constant, so in cohomology

$$E^2_{0,q} = \begin{cases} A & q = 0, \\ 0 & q \neq 0. \end{cases}$$

The cohomology of the column $p = 1$ looks like

$$E^2_{1,q} = \mathrm{H}^q(\mathrm{HR}^1(F_\bullet; A)).$$

The bicomplex spectral sequence E^0

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathrm{Hom}(\mathrm{CR}_0(F_2), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_1(F_2), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_2(F_2), A) & \xrightarrow{\partial} & \\
\mathrm{Hom}(\mathrm{CR}_0(F_1), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_1(F_1), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_2(F_1), A) & \xrightarrow{\partial} & \\
\mathrm{Hom}(\mathrm{CR}_0(F_0), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_1(F_0), A) & \xrightarrow{\partial} & \mathrm{Hom}(\mathrm{CR}_2(F_0), A) & \xrightarrow{\partial} &
\end{array}
$$

Compute the homology of the horizontal differential for E^1.

The bicomplex spectral sequence E^1

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\big\downarrow \partial & \big\downarrow \partial & \big\downarrow \partial \\
\mathrm{HR}^0(F_2; A) & \mathrm{HR}^1(F_2; A) & \mathrm{HR}^2(F_2; A) \\
\big\downarrow \partial & \big\downarrow \partial & \big\downarrow \partial \\
\mathrm{HR}^0(F_1; A) & \mathrm{HR}^1(F_1; A) & \mathrm{HR}^2(F_1; A) \\
\big\downarrow \partial & \big\downarrow \partial & \big\downarrow \partial \\
\mathrm{HR}^0(F_0; A) & \mathrm{HR}^1(F_0; A) & \mathrm{HR}^2(F_0; A)
\end{array}
$$

We know the cohomology of free racks, so we can simplify.

The bicomplex spectral sequence E^1

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\downarrow \partial & \downarrow \partial & \downarrow \partial \\
A & \mathrm{HR}^1(F_2; A) & 0 \\
\downarrow \partial & \downarrow \partial & \downarrow \partial \\
A & \mathrm{HR}^1(F_1; A) & 0 \\
\downarrow \partial & \downarrow \partial & \downarrow \partial \\
A & \mathrm{HR}^1(F_0; A) & 0
\end{array}
$$

Compute the homology of the vertical differential to get E^2.

The bicomplex spectral sequence E^2

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
 & & \\
0 & \mathrm{H}^2(\mathrm{HR}^1(F_\bullet; A)) & 0 \\
 & & \\
0 & \mathrm{H}^1(\mathrm{HR}^1(F_\bullet; A)) & 0 \\
 & & \\
A & \mathrm{H}^0(\mathrm{HR}^1(F_\bullet; A)) & 0
\end{array}
$$

We are done!

Why is this Quillen cohomology?

$$\begin{aligned}
\mathrm{D}^q(R;A) &= \mathrm{H}^q\mathrm{Hom}(\mathrm{Ab}_R(F_\bullet), R\times A)\\
&= \mathrm{H}^q\mathrm{Hom}(\mathrm{Ab}(F_\bullet), A)\\
&= \mathrm{H}^q\mathbf{Racks}(F_\bullet, A)\\
&= \mathrm{H}^q\mathrm{Hom}(\mathrm{HR}_1(F_\bullet), A)\\
&= \mathrm{H}^q\mathrm{HR}^1(F_\bullet; A)
\end{aligned}$$

Comparing the results of the computation, we get

$$\mathrm{HR}^n(R;A) = \bigoplus_{p+q=n} E^2_{p,q} = \mathrm{D}^{n-1}(R;A)$$

for $n \geqslant 1$, which is the precise version of the theorem. □

7. A handful of problems

The rack homology of quandle

The quandle homology of a free quandle is trivial but not zero.

What is the rack homology of a free quandle?

Is rack homology Ext?

There is a universal coefficient spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbf{Ab}(\mathbf{C}/X)}(\mathrm{H}_q\mathbb{L}_X(X), M) \Longrightarrow \mathrm{D}^{p+q}(X; M).$$

The edge homomorphism

$$\mathrm{Ext}^p_{\mathbf{Ab}(\mathbf{C}/X)}(\mathrm{Ab}_X(X), M) \longrightarrow \mathrm{D}^p(X; M)$$

is an isomorphism for $\mathbf{C} = \mathbf{Groups}$.

What happens for $\mathbf{C} = \mathbf{Racks}$ or $\mathbf{C} = \mathbf{Quandles}$?

The rack homology of abelian racks

The group homology of abelian groups is not trivial, but it can often be computed. In particular, the homology of a finitely generated abelian group is an exterior algebra.

What is the rack homology of $\mathbb{Z}[s, t^{\pm 1}]/(s^2 - s(1-t))$?

What is the quandle homology of $\mathbb{Z}[t^{\pm 1}]$?

How about direct sums of these?

Künneth and Mayer–Vietoris

We need more tools for computation.

Can we express $\mathrm{HR}_\bullet(R_1 \times R_2)$ in terms of the $\mathrm{HR}_\bullet(R_j)$?

Can we express $\mathrm{HR}_\bullet(R_1 \cup R_2)$ in terms of the $\mathrm{HR}_\bullet(R_j)$?

HNN extensions

If S is a rack with an automorphism φ, we can form a rack R from S by adding a new generator r such that

$$r \rhd s = \varphi(s)$$

for all s. In other words, inside R, the automorphism φ becomes inner.

What is the homology of R in terms of the homology of S and the induced action of φ on it?

Exercises and references

Exercises and references