

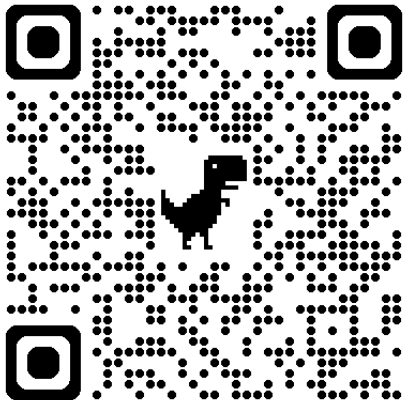
Homology of racks and quandles

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Slides for Lecture 3



6. Quillen homology

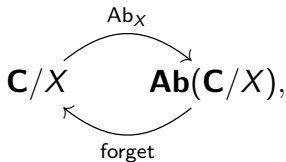
Relative abelianisation and Beck modules

\mathbf{C} = a cat like **Groups**, **Racks**, **Quandles**, **Sets**, G -**Sets**,...

$\mathbf{Ab}(\mathbf{C})$ = abelian group objects = modules over a ring

X = an object in \mathbf{C}

\mathbf{C}/X = cat of objects over $X = (Y \rightarrow X)$



The objects in $\mathbf{Ab}(\mathbf{C}/X)$ are called **Beck modules** over X .

Example: groups

$$\mathbf{Ab(\text{Groups}) = \text{Abel} = \mathbb{Z}\text{-Mod}}$$

$$\text{Ab}(G) = G/[G, G] = \mathbb{Z}G/([g \cdot h] = [g] + [h])$$

$$\mathbf{Ab(\text{Groups}/G) = \mathbb{Z}G\text{-Mod}}$$

$$\begin{aligned} (A \rightarrow G) &\longmapsto \text{Ker}(A \rightarrow G) \\ (K \rtimes G \rightarrow G) &\longleftarrow K \end{aligned}$$

$$\text{Ab}_G(G) = IG = \text{Ker}(\mathbb{Z}G \rightarrow \mathbb{Z})$$

Example: quandles and knots

$$\mathbf{Ab}(\mathbf{Quandles}) = \mathbb{Z}[t^{\pm 1}]\text{-Mod}$$

Example. $\mathbf{Ab}(Q_K) =$ (extended) Alexander module of knot K

Many non-trivial knots have trivial Alexander module...but:

Theorem (S). The **Alexander–Beck module**

$$\mathbf{Ab}_{Q_K}(Q_K)$$

detects the unknot.

It lives in an abelian category and does not involve homology!

The cotangent complex

For homology, derive the functor $Y \mapsto \text{Ab}_X(Y)$ at $Y = X$:

Choose a free resolution $F_\bullet \rightarrow X$ (so $F_\bullet \in \mathbf{sC}/X$).

Then compute the (relative) abelianisation. This, or the associated chain complex, is the **cotangent complex**

$$\mathbb{L}_X(X) = \text{Ab}_X(F_\bullet) \in \mathbf{sAb}(\mathbf{C}/X).$$

Its homology $H_\bullet \mathbb{L}_X(X)$ is the **Quillen homology** of X .

It lives in the abelian category **Ab**(\mathbf{C}/X) of Beck modules.

Example: the Quillen homology of free objects

If X is free, then the identity

$$F_{\bullet} = X$$

is a free resolution!

The space $F_{\bullet} = X$ is discrete, and so is the cotangent complex

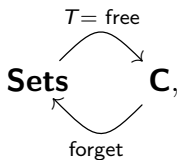
$$\mathbb{L}_X(X) = \text{Ab}_X(X).$$

Then $\text{Ab}_X(X)$ is the Quillen homology in dimension 0.

The higher Quillen homology vanishes.

Free resolutions — existence

For all our categories \mathbf{C} , we have an adjunction



and this gives free resolutions:

$$\begin{array}{c} T(X) \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \\ \leftarrow \\ \rightleftarrows \end{array} T^2(X) \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \\ \leftarrow \\ \rightleftarrows \end{array} \dots \\ \downarrow \\ X \end{array}$$

Coefficients and Quillen cohomology

We would like to replace the abelian category

$$\mathbf{Ab}(\mathbf{C}/X)$$

of Beck modules over X by the abelian category

$$\mathbf{Abel}$$

of abelian groups. It is easier and independent of \mathbf{C} and X .

Quillen **cohomology** with **coefficients** in a Beck module M

$$D^\bullet(X; M) = H^\bullet \mathrm{Hom}_{\mathbf{Ab}(\mathbf{C}/X)}(\mathbb{L}_X(X), M)$$

does just that!

From homology groups to a homology theory

Theorem (S). Rack cohomology and quandle cohomology with trivial coefficients agree with Quillen's cohomology for these categories, up to a shift.

Sketch of proof

First check this for **free** racks/quandles. The standard complex was computed by Farinati–Guccione–Guccione; the Quillen cohomology of free objects is trivial.

Then, reduce the general case to the free one using free resolutions, as before.



On the one spectral sequence — vertical first

Let $R \leftarrow F_\bullet$ be a free resolution and A an abelian group.

$$E_{p,q}^0 = \text{Hom}(\text{CR}_p(F_q), A).$$

$\text{CR}_p(R) \leftarrow \text{CR}_p(F_\bullet)$ is a free resolution of a free abelian group.

$$E_{p,q}^1 = \text{Ext}^q(\text{CR}_p(R), A) = \begin{cases} \text{Hom}(\text{CR}_p(R), A) & q = 0, \\ 0 & q \neq 0. \end{cases}$$

Then, the differential in the p -direction leaves

$$E_{p,0}^2 = \text{HR}^p(R; A)$$

in the row $q = 0$ and zeros elsewhere.

The bicomplex spectral sequence E^0

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \text{Hom}(\text{CR}_0(F_2), A) & & \text{Hom}(\text{CR}_1(F_2), A) & & \text{Hom}(\text{CR}_2(F_2), A) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \text{Hom}(\text{CR}_0(F_1), A) & & \text{Hom}(\text{CR}_1(F_1), A) & & \text{Hom}(\text{CR}_2(F_1), A) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \text{Hom}(\text{CR}_0(F_0), A) & & \text{Hom}(\text{CR}_1(F_0), A) & & \text{Hom}(\text{CR}_2(F_0), A) \end{array}$$

Compute the homology of the vertical differential to get E^1 .

The bicomplex spectral sequence E^1

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \end{array}$$

$$\mathrm{Hom}(\mathrm{CR}_0(R), A) \xrightarrow{\partial} \mathrm{Hom}(\mathrm{CR}_1(R), A) \xrightarrow{\partial} \mathrm{Hom}(\mathrm{CR}_2(R), A)$$

Compute the homology of the horizontal differential for E^2 .

The bicomplex spectral sequence E^2

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \text{HR}^0(R; A) & \text{HR}^1(R; A) & \text{HR}^2(R; A) \end{array}$$

We are done!

On the other spectral sequence — horizontal first

$$E_{p,q}^0 = \text{Hom}(\text{CR}_p(F_q), A)$$

First, the homology in the p -direction:

$$E_{p,q}^1 = \text{HR}^p(F_q; A)$$

This is non-zero only in the two columns $p = 0$ and $p = 1$.

The column $p = 0$ is constant, so in cohomology

$$E_{0,q}^2 = \begin{cases} A & q = 0, \\ 0 & q \neq 0. \end{cases}$$

The cohomology of the column $p = 1$ looks like

$$E_{1,q}^2 = H^q(\text{HR}^1(F_\bullet; A)).$$

The bicomplex spectral sequence E^0

 \vdots \vdots \vdots

$$\text{Hom}(\text{CR}_0(F_2), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_1(F_2), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_2(F_2), A) \xrightarrow{\partial}$$

$$\text{Hom}(\text{CR}_0(F_1), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_1(F_1), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_2(F_1), A) \xrightarrow{\partial}$$

$$\text{Hom}(\text{CR}_0(F_0), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_1(F_0), A) \xrightarrow{\partial} \text{Hom}(\text{CR}_2(F_0), A) \xrightarrow{\partial}$$

Compute the homology of the horizontal differential for E^1 .

The bicomplex spectral sequence E^1

$$\begin{array}{ccccc} & \vdots & & \vdots & & \vdots \\ & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ \text{HR}^0(F_2; A) & & \text{HR}^1(F_2; A) & & \text{HR}^2(F_2; A) \\ & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ \text{HR}^0(F_1; A) & & \text{HR}^1(F_1; A) & & \text{HR}^2(F_1; A) \\ & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ \text{HR}^0(F_0; A) & & \text{HR}^1(F_0; A) & & \text{HR}^2(F_0; A) \end{array}$$

We know the cohomology of free racks, so we can simplify.

The bicomplex spectral sequence E^1

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ A & \text{HR}^1(F_2; A) & & 0 & \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ A & \text{HR}^1(F_1; A) & & 0 & \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ A & \text{HR}^1(F_0; A) & & 0 & \end{array}$$

Compute the homology of the vertical differential to get E^2 .

The bicomplex spectral sequence E^2

$$\begin{array}{ccccc} & \vdots & & \vdots & & \vdots \\ & & & & & \\ 0 & & H^2(\mathrm{HR}^1(F_\bullet; A)) & & 0 \\ & & & & \\ 0 & & H^1(\mathrm{HR}^1(F_\bullet; A)) & & 0 \\ & & & & \\ A & & H^0(\mathrm{HR}^1(F_\bullet; A)) & & 0 \end{array}$$

We are done!

Why is this Quillen cohomology?

$$\begin{aligned}D^q(R; A) &= H^q \text{Hom}(\text{Ab}_R(F_\bullet), R \times A) \\&= H^q \text{Hom}(\text{Ab}(F_\bullet), A) \\&= H^q \mathbf{Racks}(F_\bullet, A) \\&= H^q \text{Hom}(\text{HR}_1(F_\bullet), A) \\&= H^q \text{HR}^1(F_\bullet; A)\end{aligned}$$

Comparing the results of the computation, we get

$$\text{HR}^n(R; A) = \bigoplus_{p+q=n} E_{p,q}^2 = D^{n-1}(R; A)$$

for $n \geq 1$, which is the precise version of the theorem. □

7. A handful of problems

The rack homology of quandle

The quandle homology of a free quandle is trivial but not zero.

What is the rack homology of a free quandle?

Is rack homology Ext?

There is a universal coefficient spectral sequence

$$E_2^{p,q} = \text{Ext}_{\mathbf{Ab}(\mathbf{C}/X)}^p(H_q \mathbb{L}_X(X), M) \implies D^{p+q}(X; M).$$

The edge homomorphism

$$\text{Ext}_{\mathbf{Ab}(\mathbf{C}/X)}^p(\text{Ab}_X(X), M) \longrightarrow D^p(X; M)$$

is an isomorphism for $\mathbf{C} = \mathbf{Groups}$.

What happens for $\mathbf{C} = \mathbf{Racks}$ or $\mathbf{C} = \mathbf{Quandles}$?

The rack homology of abelian racks

The group homology of abelian groups is not trivial, but it can often be computed. In particular, the homology of a finitely generated abelian group is an exterior algebra.

What is the rack homology of $\mathbb{Z}[s, t^{\pm 1}]/(s^2 - s(1 - t))$?

What is the quandle homology of $\mathbb{Z}[t^{\pm 1}]$?

How about direct sums of these?

Künneth and Mayer–Vietoris

We need more tools for computation.

Can we express $HR_{\bullet}(R_1 \times R_2)$ in terms of the $HR_{\bullet}(R_j)$?

Can we express $HR_{\bullet}(R_1 \cup R_2)$ in terms of the $HR_{\bullet}(R_j)$?

HNN extensions

If S is a rack with an automorphism φ , we can form a rack R from S by adding a new generator r such that

$$r \triangleright s = \varphi(s)$$

for all s . In other words, inside R , the automorphism φ becomes inner.

What is the homology of R in terms of the homology of S and the induced action of φ on it?

Exercises and references

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