Homology of racks and quandles

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Slides for Lectures 1 and 2

4. Homology of racks and quandles

Rack homology

$R =$ a rack

$$\mathrm{CR}_n(R) = \mathbb{Z}(R^n) = (\mathbb{Z}R)^{\otimes n}$$

$$\begin{aligned} A_j(x_1, \dots, x_n) &= (x_1, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_n) \\ B_j(x_1, \dots, x_n) &= (x_1, \dots, x_{j-1}, \widehat{x_j}, x_j \triangleright x_{j+1}, \dots, x_j \triangleright x_n) \end{aligned}$$

$$\partial = \sum_{j=1}^{n} (-1)^j (A_j - B_j)$$

This defines (!) a chain complex.

$$\mathrm{HR}_\bullet(R) = \mathrm{H}(\mathrm{CR}_\bullet(R))$$

The first differential

Note that

$$\mathsf{CR}_\bullet(R) = \mathsf{T}R$$

is the tensor algebra on R. The first differential

$$\partial\colon \mathbb{Z}R \longrightarrow \mathbb{Z}$$

is given by $\partial(x) = 0$. It follows that

$$\mathsf{HR}_0(R) = \mathbb{Z}.$$

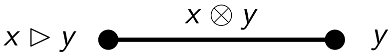
Note ∂ is **not** a derivation.

The second differential

The second differential

$$\partial\colon \mathbb{Z}R \otimes \mathbb{Z}R \longrightarrow \mathbb{Z}R$$

is given by $\partial(x \otimes y) = y - x \triangleright y$.



This implies that

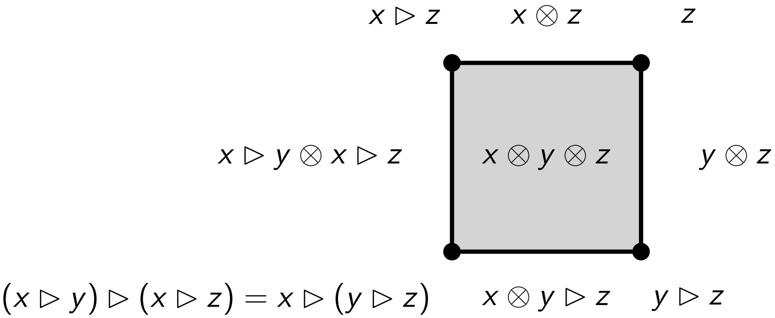
$$\mathrm{HR}_1(R) = \mathbb{Z}(R/\mathrm{Gr}(R))$$

is the free abelian group with basis the orbits of R.

The third differential

$$\partial(x \otimes y \otimes z) = (y \otimes z - (x \triangleright y \otimes x \triangleright z)) - x \otimes (z - y \triangleright z)$$

$\partial^2(x \otimes y \otimes z) = 0$:



$x \triangleright z$ $\quad$ $x \otimes z$ $\quad$ z

$x \triangleright y \otimes x \triangleright z$ $\quad$ $x \otimes y \otimes z$ $\quad$ $y \otimes z$

$(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ $\quad$ $x \otimes y \triangleright z$ $\quad$ $y \triangleright z$

Extensions

Let R be a rack and A be an abelian group.

A map $\varphi\colon R \times R \to A$ defines a rack structure on $R \times A$ via

$$(x, i) \rhd (y, j) = (x \rhd y, j + \varphi(x, y))$$

if and only if

$$\varphi(x \rhd y, x \rhd z) + \varphi(x, z) = \varphi(x, y \rhd z) + \varphi(y, z).$$

In words, the map φ must be a 2–cocycle with values in A.

$\mathrm{HR}^2(R; A) = \mathrm{H}^2\mathrm{Hom}(\mathrm{CR}_\bullet(R), A)$ classifies extensions

The homology of trivial racks

If (X, id) is a trivial (permutation) rack, we have $\partial = 0$, so

$$\mathrm{HR}_\bullet(X, \mathrm{id}) = \mathrm{T}X$$

is the tensor algebra generated by X.

In particular, if $X = \star$ is a singleton, then

$$\mathrm{HR}_n(\star) = \mathbb{Z}$$

for all non-negative n.

Conjugation is trivial

Recall: $c_g\colon G \to G, x \mapsto g \triangleright x = gxg^{-1}$ induces

$$\mathrm{H}_\bullet(c_g) = \mathrm{id} \quad \text{on} \quad \mathrm{H}_\bullet(G).$$

Proposition. $l_r\colon R \to R, x \mapsto r \triangleright x$ induces

$$\mathrm{HR}_\bullet(l_g) = \mathrm{id} \quad \text{on} \quad \mathrm{HR}_\bullet(R).$$

Proof.

$$H(x_1, \ldots, x_n) = (r, x_1, \ldots, x_n)$$

$$(\partial H + H\partial)(x_1, \ldots, x_n) = (r \triangleright x_1, \ldots, r \triangleright x_n) - (x_1, \ldots, x_n)$$

□

Rack homology with coefficients

Let R be a rack and S a $\mathrm{Gr}(R)$–set (case before: $S = \star$).

$$\mathrm{CR}_n(R; S) = \mathbb{Z}(R^n \times S) = (\mathbb{Z}R)^{\otimes n} \otimes \mathbb{Z}S$$

$$A_j(x_1, \dots, x_n, s) = (x_1, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_n, s)$$
$$B_j(x_1, \dots, x_n, s) = (x_1, \dots, x_{j-1}, \widehat{x_j}, x_j \triangleright x_{j+1}, \dots, x_j \triangleright x_n, x_j \cdot s)$$

$$\partial = \sum_{j=1}^{n} (-1)^j (A_j - B_j)$$

This defines (!) a chain complex.

$$\mathrm{HR}_\bullet(R; S) = \mathrm{H}(\mathrm{CR}_\bullet(R; S))$$

Other coefficients

The most important cases are $S = \star$, $S = R$, and $S = \mathsf{Gr}(R)$.

These are related by the canonical maps

$$R \longrightarrow \mathsf{Gr}(R) \longrightarrow \star$$

of $\mathsf{Gr}(R)$–sets.

The identity map gives $\mathsf{CR}_n(R; R) = \mathsf{CR}_{n+1}(R; \star)$.

$$\mathsf{HR}_n(R; \star) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathsf{HR}_{n-1}(R; R) & n > 0. \end{cases}$$

The pictures before were for $\mathsf{CR}_n(R; R)$!

Multiplicative and comultiplicative structure

Clauwens: There are homomorphisms

$$\mathrm{CR}_\bullet(R;\mathrm{Gr}(R)) \otimes \mathrm{CR}_\bullet(R;S) \to \mathrm{CR}_\bullet(R;S)$$

given by

$$(x_\bullet, g) \otimes (y_\bullet, s) \mapsto (x_\bullet, g \cdot y_\bullet, g \cdot s).$$

$\mathrm{CR}_\bullet(R;\mathrm{Gr}(R))$ is a differential graded algebra

$\mathrm{CR}_\bullet(R;S)$ is a differential graded module over it

Covez–Farinati–Lebed–Manchon: differential graded bialgebra

Homology of quandles

Q = a quandle

$\mathrm{CD}_\bullet(Q)$ = the sub(!)complex of $\mathrm{CR}_\bullet(Q)$ with

$$\{(x_1, ..., x_n) \mid x_j = x_{j+1} \text{ for at least one } j\}$$

$\mathrm{CQ}_\bullet(Q)$ = the quotient complex

$$0 \longrightarrow \mathrm{CD}_\bullet(Q) \longrightarrow \mathrm{CR}_\bullet(Q) \longrightarrow \mathrm{CQ}_\bullet(Q) \longrightarrow 0$$

Litherland–Nelson: That short exact sequence splits.

$$\mathrm{HQ}_n(Q) = \mathrm{H}(\mathrm{CQ}_\bullet(Q)) = \textbf{quandle homology} \text{ of } Q$$

Przytycki–Putyra: This determines the rack homology.

The homology of free racks and quandles

Theorem (Farinati–Guccione–Guccione). S a set

$$\mathrm{HR}_n(\mathrm{FR}(S);\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{\oplus S} & n = 1 \\ 0 & n \geqslant 2 \end{cases}$$

A similar result holds for quandles.

A somewhat simpler argument is given in [Lebed–S, Rem. 2.6], but both use the standard complex.

This should not be a theorem, but the definition!

Betti numbers and torsion

The rational homology of a finite group G is trivial:

$$\mathrm{H}_n(G;\mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

In contrast, the rational homology of a finite rack R is big:

Theorem (Etingof–Grana, Litherland–Nelson).

$$\mathrm{HR}_\bullet(R;\mathbb{Q}) = \mathrm{T}(R/\mathrm{Gr}(R)) \otimes \mathbb{Q}.$$

Both papers have information on the torsion that can occur.

The homology of prime order quandles

$p =$ a prime and $x \triangleright_t y = tx + (1-t)y$ for $t \notin \{0,1\}$

$e =$ multiplicative order of t

$$b(1) = b(2) = \cdots = b(2e-2) = 0$$

$$b(2e-1) = b(2e) = 1$$

$$b(n+2e) = b(n) + b(n+1) + b(n+2)$$

Theorem (Clauwens for $t = -1$, Nosaka in general).

$$\mathrm{HQ}_n(\mathbb{Z}/p, \triangleright_t) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus (\mathbb{Z}/p)^{\oplus b(1)} & n = 1 \\ (\mathbb{Z}/p)^{\oplus b(n)} & n \geqslant 2 \end{cases}$$

5. Permutation racks

The homology of permutation racks

$x \triangleright y = \varphi(y)$ for a permutation $\varphi\colon X \to X$

r = number of orbits = $|X/\varphi|$

r_{fin} = number of finite orbits

Theorem (Lebed–S). The homology of (X, φ) is a free abelian group. In particular, it is torsion-free.

$$\sum_{n=0}^{\infty} \operatorname{rank}(\mathrm{HR}_n) t^n = \frac{1+t}{1-(r-1)t - r_{\text{fin}} t^2}$$

The homology of finite permutation racks

Finite permutation racks have $r_{\text{fin}} = r$, so

$$\frac{1+t}{1-(r-1)t-r_{\text{fin}}t^2} = \frac{1+t}{(1-rt)(1+t)} = \frac{1}{1-rt}.$$

Corollary (Lebed–S). If X is finite, then

$$\mathrm{HR}_n(X/\varphi) = \mathbb{Z}(X/\varphi)^{\otimes n}.$$

is the tensor algebra on the set X/φ of orbits.

Our proof requires infinite permutation racks, as we are using free resolutions, and the free ones are infinite.

The homology of free permutation racks

Free permutation racks have $r_{\text{fin}} = 0$, so

$$\frac{1+t}{1-(r-1)t-r_{\text{fin}}t^2} = \frac{1+t}{1-(r-1)t}.$$

Corollary (Lebed–S). If $X = \mathbb{Z} \times S$ is free, then

$$\mathrm{HR}_n(\mathbb{Z} \times S) = \overline{\mathbb{Z}S}^{\otimes(n-1)} \otimes \mathbb{Z}S.$$

$\overline{\mathbb{Z}S}$ = linear combinations with zero coefficient-sum

This follows from the main result, but it is actually used as an ingredient: it describes $\mathrm{HR}_n(\mathbb{Z} \times S)$ as a nice functor of S.

Outline of proof

First, do **free** permutation racks $X = \mathbb{Z} \times S$ and express the result as a functor of the set S of orbits.

Resolve a general X by free ones, for example $\mathbb{R} \times X \xrightarrow{\sim} X$. The resolution gives a **spectral sequence**.

The orbits of $\mathbb{R} \times X$ are the **homotopy orbits** of X, and these are points (for the free orbits) and circles (for the finite orbits). This describes the E_2 page.

The spectral sequence **degenerates**. $\square$

The spectral sequence

Proposition (Lebed–S). For every permutation rack, there is a spectral sequence with

$$E^2_{p,\bullet} = \overline{\mathrm{H}}_\bullet(X /\!\!/ \varphi)^{\otimes(p-1)} \otimes \mathrm{H}_\bullet(X /\!\!/ \varphi)$$

and converging to the rack homology $\mathrm{HR}_\bullet(X,\varphi)$.

Start with a resolution $X \leftarrow F_\bullet$ by free permutation racks and apply the functor $\mathrm{CR}_\bullet(-)$. This gives a bicomplex

$$E^0_{p,q} = \mathrm{CR}_p(F_q).$$

The bicomplex $E^0_{p,q} = \mathsf{CR}_p(F_q)$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathsf{CR}_0(F_2) \arrow[d] & \mathsf{CR}_1(F_2) \arrow[l] \arrow[d] & \mathsf{CR}_2(F_2) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathsf{CR}_0(F_1) \arrow[d] & \mathsf{CR}_1(F_1) \arrow[l] \arrow[d] & \mathsf{CR}_2(F_1) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathsf{CR}_0(F_0) & \mathsf{CR}_1(F_0) \arrow[l] & \mathsf{CR}_2(F_0) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Bicomplexes

$$\begin{tikzcd}
\vdots \arrow[d, "\partial_{\mathsf{v}}"'] & \vdots \arrow[d, "\partial_{\mathsf{v}}"'] & \vdots \arrow[d, "\partial_{\mathsf{v}}"'] & \\
E^0_{0,2} \arrow[d, "\partial_{\mathsf{v}}"'] & E^0_{1,2} \arrow[l, "\partial_{\mathsf{h}}"'] \arrow[d, "\partial_{\mathsf{v}}"'] & E^0_{2,2} \arrow[l, "\partial_{\mathsf{h}}"'] \arrow[d, "\partial_{\mathsf{v}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"'] \\
E^0_{0,1} \arrow[d, "\partial_{\mathsf{v}}"'] & E^0_{1,1} \arrow[l, "\partial_{\mathsf{h}}"'] \arrow[d, "\partial_{\mathsf{v}}"'] & E^0_{2,1} \arrow[l, "\partial_{\mathsf{h}}"'] \arrow[d, "\partial_{\mathsf{v}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"'] \\
E^0_{0,0} & E^0_{1,0} \arrow[l, "\partial_{\mathsf{h}}"'] & E^0_{2,0} \arrow[l, "\partial_{\mathsf{h}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"']
\end{tikzcd}$$

$$(\partial_{\mathsf{v}})^2 = 0 \qquad (\partial_{\mathsf{h}})^2 = 0 \qquad \partial_{\mathsf{v}}\partial_{\mathsf{h}} = \pm\partial_{\mathsf{h}}\partial_{\mathsf{v}}$$

The bicomplex spectral sequence E^0

$$\begin{tikzcd}
\vdots \arrow[d, "\partial_v"'] & \vdots \arrow[d, "\partial_v"'] & \vdots \arrow[d, "\partial_v"'] & \\
E^0_{0,2} \arrow[d, "\partial_v"'] & E^0_{1,2} \arrow[d, "\partial_v"'] & E^0_{2,2} \arrow[d, "\partial_v"'] & \cdots \\
E^0_{0,1} \arrow[d, "\partial_v"'] & E^0_{1,1} \arrow[d, "\partial_v"'] & E^0_{2,1} \arrow[d, "\partial_v"'] & \cdots \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & \cdots
\end{tikzcd}$$

Compute the homology of the vertical differential to get E^1.

The bicomplex spectral sequence E^1

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "\partial_{\mathsf{h}}"'] & E^1_{2,2} \arrow[l, "\partial_{\mathsf{h}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"'] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "\partial_{\mathsf{h}}"'] & E^1_{2,1} \arrow[l, "\partial_{\mathsf{h}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"'] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "\partial_{\mathsf{h}}"'] & E^1_{2,0} \arrow[l, "\partial_{\mathsf{h}}"'] & \cdots \arrow[l, "\partial_{\mathsf{h}}"']
\end{tikzcd}$$

Compute the homology of the horizontal differential for E^2.

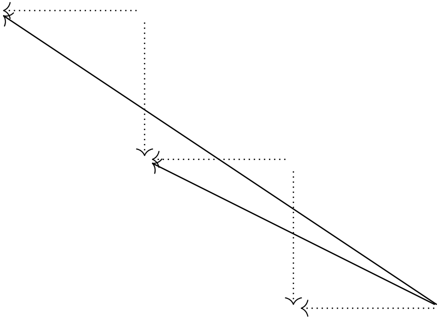
The bicomplex spectral sequence E^2

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
 & & & \\
E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & \cdots \\
 & & & \\
E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & \cdots \\
 & & & \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & \cdots
\end{array}$$

We are **not** done! There is a ∂^2 on E^2, a ∂^3 on E^3, ...

Staircases

The higher differentials are given by choosing witnesses for the fact that the lower differentials are zero in homology.



∂^2 and ∂^3

The other bicomplex spectral sequence

The bicomplex spectral sequence computes the homology of the chain complex with

$$C_n = \bigoplus_{p+q=n} E^0_{p,q}$$

and

$$\partial = \partial_{\mathsf{v}} \pm \partial_{\mathsf{h}}.$$

As this is symmetric in p and q, the **other** bicomplex spectral sequence, obtained by interchanging ∂_{v} and ∂_{h}, computes the same thing!

Back to $E^0_{p,q} = \mathrm{CR}_p(F_q)$

First, compute the differential in the q–direction:

$$E^1_{p,q} = \begin{cases} \mathrm{CR}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Now compute the differential in the p–direction:

$$E^2_{p,q} = \begin{cases} \mathrm{HR}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

All other differentials are zero, so this spectral sequence is **not** the one in the Proposition, which is the other one. But this one shows what it computes: the rack homology of X.

The other spectral sequence

First, compute the differential in the p–direction:

$$\begin{aligned} E^1_{p,q} &= \mathrm{HR}_p(F_q) \\ &= \overline{\mathbb{Z}(F_q/\varphi)}^{\otimes(p-1)} \otimes \mathbb{Z}(F_q/\varphi) \end{aligned}$$

The homology of a **free** permutation rack is given by orbits.

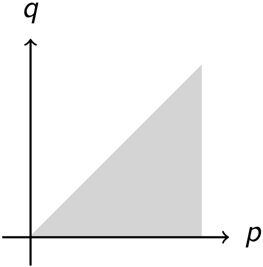
Now compute the differential in the q–direction:

$$E^2_{p,\bullet} = \overline{\mathrm{H}}_\bullet(X /\!\!/ \varphi)^{\otimes(p-1)} \otimes \mathrm{H}_\bullet(X /\!\!/ \varphi).$$

The orbits of $F_\bullet$ are the homotopy orbits of X.

The differentials

$$E^2_{p,\bullet} = \overline{\mathrm{H}}_\bullet(X /\!\!/ \varphi)^{\otimes(p-1)} \otimes \mathrm{H}_\bullet(X /\!\!/ \varphi)$$



This gives an upper bound on the rack homology. We produce enough classes to show that the bound is sharp, so nothing can be cancelled by differentials.

Proposition (Lebed–S). The spectral sequence collapses on the second page.

Exercises and references

Exercises and references