Homology of racks and quandles

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Slides for Lecture 1

2. Homological algebra

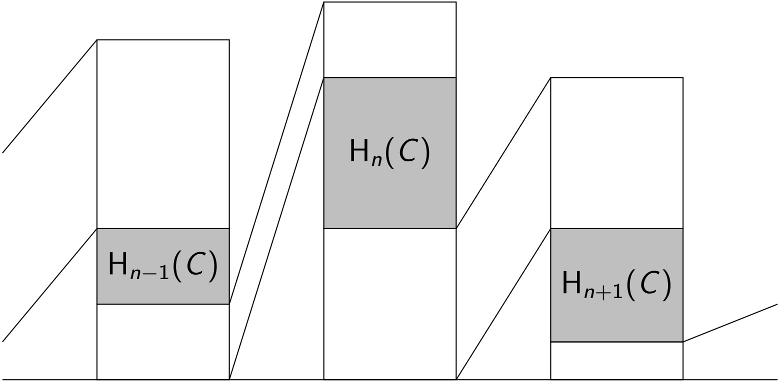
Chain complexes are cochain complexes

$$\begin{tikzcd}
\cdots & C_{-1} \arrow[l, "\partial"'] \arrow[d, equal] & C_0 \arrow[l, "\partial"'] \arrow[d, equal] & C_1 \arrow[l, "\partial"'] \arrow[d, equal] & \cdots \arrow[l, "\partial"'] \\
\cdots & C^1 \arrow[l, "\partial"'] & C^0 \arrow[l, "\partial"'] & C^{-1} \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"']
\end{tikzcd}$$

$$\partial^2 = 0 \implies \mathrm{Im}(\partial) \leqslant \mathrm{Ker}(\partial)$$

$$\begin{aligned} \text{(co)homology } &= \text{cycles / boundaries} \\ \mathrm{H}(C, \partial) &= \mathrm{Ker}(\partial) \;/\; \mathrm{Im}(\partial) \end{aligned}$$

Homology



$$\xleftarrow{\partial} C_{n-1} \xleftarrow{\partial} C_n \xleftarrow{\partial} C_{n+1} \xleftarrow{\partial}$$

Homomorphisms and homotopies

C,D : complexes

define a new complex $\mathrm{HOM}(C, D)$:

$$\mathrm{HOM}(C, D)^n = \prod_m \mathrm{Hom}(C^m, D^{m+n})$$
$$(\partial f)c = \partial(fc) - (-1)^{|f|} f(\partial c).$$

$$\begin{aligned}
\mathrm{Ker}(\partial)^0 &= \text{homomorphism } C \longrightarrow D \\
\mathrm{Im}(\partial)^0 &= \text{nullhomotopic homs } C \longrightarrow D \\
\mathrm{H}^0\mathrm{HOM}(C, D) &= \text{homotopy classes of homs } C \longrightarrow D
\end{aligned}$$

Free resolutions, Ext and Tor

A **free resolution** of an R–module M is a homomorphism

$$
\begin{array}{c}
P \\
\downarrow e \\
M
\end{array}
\qquad
\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & P_0 \arrow[l] \arrow[d, "e"'] & P_1 \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & 0 \arrow[l] & M \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

such that all P_n are free and $\mathrm{H}(e)$ an iso (e an **equivalence**).

$$
\begin{aligned}
\mathrm{Ext}^n_R(M,N) &= \mathrm{H}^n(\mathrm{Hom}(P,N)) \\
\mathrm{Tor}^R_n(M,N) &= \mathrm{H}_n(P \otimes_R N))
\end{aligned}
$$

Example: the (co)homology of groups

For $R = \mathbb{Z}G$ the group ring, M a module over it:

$$\begin{aligned} \mathsf{H}_n(G; N) &= \mathsf{Tor}^n_{\mathbb{Z}G}(\mathbb{Z}, N), \\ \mathsf{H}^n(G; N) &= \mathsf{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, N). \end{aligned}$$

This should not be the definition, but a theorem!

How to find free resolutions of the trivial $\mathbb{Z}G$–module $\mathbb{Z}$?

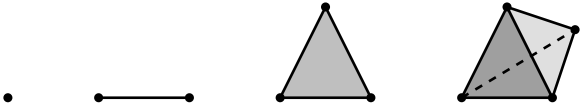
3. Algebraic topology

Spaces

There is a combinatorial model for spaces that treats them as 'non-linear chain complexes': **simplicial sets**.

$$
X \qquad\qquad
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[l, shift left=3, "\partial_0"'] \arrow[l, shift right=3, "\partial_1"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & X_2 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=4.5] \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[l, shift right=4.5]
\end{tikzcd}
$$

$$X_n = \{n\text{–simplices in } X\} = \{\Delta^n \to X\}$$



Kan–Quillen

There is an adjunction

$$\begin{tikzcd} \mathbf{sSet} \arrow[rr, bend left, "|-|"] & & \mathbf{Top} \arrow[ll, bend left, "\mathrm{Sing}(-)"] \end{tikzcd}$$

given by

$$|X| = \left(\coprod_n X_n \times \Delta^n_{\mathrm{top}} \right) / \sim$$

and

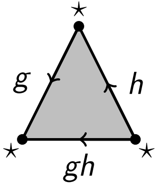
$$\mathrm{Sing}(Y)_n = \mathbf{Top}(\Delta^n_{\mathrm{top}}, Y).$$

Theorem (Kan–Quillen). This adjunction induces an equivalence of homotopy theories.

Classifying spaces...

G : group (or monoid)

$$\mathrm{B}G \qquad \star \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \times G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$



$\pi_1(\mathrm{B}G, \star) = G$

What is the universal cover?

...and their universal covers

G : group (or monoid)

$$\mathrm{E}G \qquad G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \times G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \times G \times G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

$$\begin{tikzcd} & hx \arrow[dl, "g"'] & \\ ghx & & x \arrow[ul, "h"'] \arrow[ll, "gh"] \end{tikzcd}$$

$\mathrm{E}G \simeq \star$ with free G–action

Examples. $\mathrm{B}\mathbb{Z} \simeq \mathrm{S}^1$ and $\mathrm{E}\mathbb{Z} \simeq \mathbb{R}$

Abelian spaces are simplicial abelian groups

$$\mathbf{Ab}(\mathbf{sSet}) = \mathbf{sAbel}$$

$$A \qquad\qquad A_0 \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} A_1 \underset{\partial_2}{\overset{\partial_0}{\Lleftarrow}} A_2 \cdots$$

$$\partial = \sum_j (-1)^j \partial_j \quad \Longrightarrow \quad \partial^2 = 0$$

Theorem (Dold–Kan). The category **sAbel** is equivalent to the category of non-negative chain complexes.

$$\pi_n(|A|, 0) = \mathrm{H}_n(A, \partial)$$

Homology is abelianisation

$$\begin{tikzcd}
X : \mathbf{Top} \arrow[rr, dashed] \arrow[d, "\sim"'] & & (\mathbb{Z}\mathsf{Sing}(X), \partial) : \mathbf{Chain} \\
\mathsf{Sing}(X) : \mathbf{sSet} \arrow[rr, "\mathbb{Z}-"'] & & \mathbb{Z}\mathsf{Sing}(X) : \mathbf{sAbel} \arrow[u, "\sim"']
\end{tikzcd}$$

Application.

$\mathbb{Z}\mathsf{E}G$ is a free resolution of the G–module $\mathbb{Z}\star = \mathbb{Z}$

$\mathsf{H}_*(G) = \mathsf{H}_*(\mathsf{B}G)$

Homotopy orbit spaces

$X =$ a G–set (or G–space)

$X/G =$ the set (or space) of orbits

Replace X by a free resolution:

$$\mathrm{E}G \times X \xrightarrow{\sim} X.$$

This is a space even if X is a set! Its orbit space

$$X/\!\!/G = (\mathrm{E}G \times X)/G = \mathrm{E}G \times_G X$$

is the **homotopy orbit space** of X.

Mapping tori

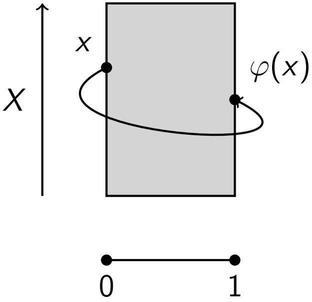
$G = \mathbb{Z}$

$\mathbb{Z}$–space $X = \varphi\colon X \xrightarrow{\cong} X$

$\mathbb{R} \times X \xrightarrow{\sim} X$: free resolution

$X /\!\!/ \mathbb{Z} = \mathbb{R} \times_{\mathbb{Z}} X$

mapping torus of φ



$X =$ free orbit: $X /\!\!/ \mathbb{Z} \simeq \star$

$X =$ finite orbit: $X /\!\!/ \mathbb{Z} \simeq \mathsf{S}^1$

Exercises and references

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