Homology of racks and quandles

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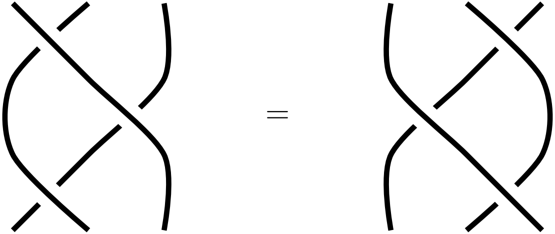
1. Racks and quandles

Braids and the Yang–Baxter equation

$$(\beta \times \mathrm{id})(\mathrm{id} \times \beta)(\beta \times \mathrm{id}) = (\mathrm{id} \times \beta)(\beta \times \mathrm{id})(\mathrm{id} \times \beta)$$



A map $\beta \colon X \times X \to X \times X,\ (x, y) \mapsto (x \triangleright y, x)$ induces actions of the braid groups B_n on the powers X^n of X if and only if

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

Abstraction

對稱變換ノ抽象化

(圭ノ理論ノ序說)

高崎光久（哈爾濱）

Abstraction of symmetric transformations.

By Mituhisa TAKASAKI, Harpin.

概　　要

（一） 先ヅ簡單ナ四個ノ公理カラ，圭及ビ圭ノ演算（對稱變換）$ab=c$ ガ定義サレル.（第一章）. ソレハ點對稱變換，直線對稱變換，或ル反形法等ノ抽象化ニ當リ（第三章），一般ニ結合，交換ノ兩法則ヲ持タズ，本來如何ナル特別元素ヲモ含マズ，マタ $ax=b$ ノ解 x ノ個數ニ就テ何等ノ制限ガ設ケラレテ居ナイノヲ特徵トスル.

[...]

ヨリ大キイ次數ニハナレナイ.　(総)

3. （定義） 任意三元素ガ公式

$$a(bc)=(ab)(ac) \qquad (卯)$$

ヲ滿足スル圭ヲ文圭ト呼ブ.

本圭ガスベテ文圭デアルコトハ明ラカデアル. ナホ文圭ニ就テハ次ノ諸項ガ知ラレル.

Racks

A **rack** $(R, \triangleright)$ is a set R together with a binary operation

$$\triangleright\colon R \times R \longrightarrow R$$

such that all left-multiplications are bijections and satisfy

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

Equivalently, the left-multiplications are homomorphisms.

As they are bijective, they are automorphisms.

Dates and names

1880	Peirce
1943	Takasaki
1959	Conway–Wraith
1967	Loos
1978	Burde
1982	Joyce, Matveev
⋮	⋮

Permutations and racks

If (X, φ) is a set X with a permutation $\varphi\colon X \to X$, then X becomes a rack via

$$x \triangleright y = \varphi(y).$$

These are the **permutation racks**.

For $\varphi = \mathrm{id}_X$, we get the **trivial** racks.

If $(R, \triangleright)$ is a rack, then

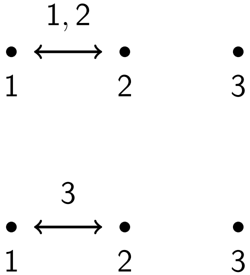
$$\sigma(x) = x \triangleright x$$

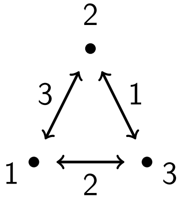
defines an automorphism of $(R, \triangleright)$.

For a permutation rack (X, φ), we have $\sigma = \varphi$.

Six racks with three elements up to isomorphism

Three permutation racks: id, $(1,2)$, $(1,2,3)$, and:





Symmetry...

...is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.

Hermann Weyl (1885–1955)

Groups, conjugation, and quandles

$(G, \cdot) =$ a group

$$x \triangleright y = xyx^{-1}$$

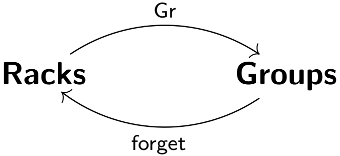
$(G, \triangleright) =$ a rack

We can replace G by any subset closed under conjugation.

We have $x \triangleright x = x$ for these racks.

A **quandle** is a rack R with $x \triangleright x = x$ for all x.

Enveloping groups



Universal property:

$$\mathbf{Groups}(\mathsf{Gr}(R), G) \cong \mathbf{Racks}(R, G)$$

Model:

$$\mathsf{Gr}(R) = \big\langle g(x), x \in R \mid g(x \rhd y) = g(x)g(y)g(x)^{-1} \big\rangle.$$

Examples. If (X, φ) is a permutation rack, then $\mathsf{Gr}(X, \varphi)$ is free abelian (!) with basis the set X/φ of orbits.

Braids and Yetter–Drinfeld sets

A **crossed** G–**set** is a G–set X together with a G–map

$$c\colon X \longrightarrow \mathrm{Ad}(G),$$

where $\mathrm{Ad}(G)$ is the G–action on itself via conjugation.

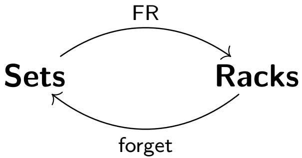
A crossed G–set defines a rack structure via

$$x \rhd y = c(x) \cdot y.$$

Every rack R has this form: take $X = R$ and $G = \mathrm{Gr}(R)$.

The category of crossed G–sets is braided; it is the Drinfeld centre of the category of G–sets.

Free racks



Universal property:

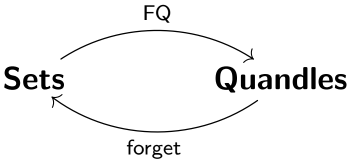
$$\mathbf{Racks}(\mathsf{FR}(S), R) \cong \mathbf{Sets}(S, R)$$

Model:

$$\mathsf{FR}(S) = \mathsf{F}(S) \times S \text{ where } \mathsf{F}(S) = \text{free group on } S$$

$$(g, s) \rhd (h, t) = (gsg^{-1}h, t)$$

Free quandles



Universal property:

$$\mathbf{Quandles}(\mathsf{FQ}(S), R) \cong \mathbf{Sets}(S, R)$$

Model:

$$\mathsf{FQ}(S) = \{gsg^{-1} \in \mathsf{F}(S) \mid g \in G \text{ and } s \in S\}$$

$$\rhd = \text{ conjugation}$$

The automorphism groups

$$\begin{aligned}\mathsf{Aut}(\mathsf{FQ}(S)) &= \{\Phi \in \mathsf{Aut}(\mathsf{F}(S)) \mid \Phi(S) \text{ is conjugate to } S\} \\ &= \langle \text{symmetric group}, \text{braid group (Artin)} \rangle\end{aligned}$$

$$\mathsf{Aut}(\mathsf{FR}(S)) = \mathbb{Z}S \rtimes \mathsf{Aut}(\mathsf{FQ}(S))$$

As $\mathsf{Aut}(\mathsf{FR}(S))$ acts on $\mathsf{FR}(S)$, this gives another reason why the braid groups act on

$$\mathbf{Sets}(S, R) = \mathbf{Racks}(\mathsf{FR}(S), R).$$

Loos, O.
Math. Zeitschr. 99, 141 — 170 (1967)

Spiegelungsräume und homogene symmetrische Räume*

Ottmar Loos

Eingegangen am 22. August 1966

Einleitung

Ein *Spiegelungsraum* ist eine Menge M mit einer Multiplikation $\mu: M \times M \to M$, $(x, y) \mapsto x \cdot y$, die den folgenden Axiomen genügt:

(S1) $$x \cdot x = x,$$

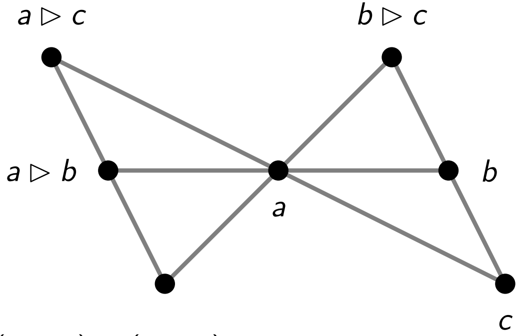
(S2) $$x \cdot (x \cdot y) = y,$$

(S3) $$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z).$$

Es sei $S(x): y \mapsto x \cdot y$ die Linksmultiplikation mit x in M. Dies ist eine involutorische Abbildung von M auf sich, die den Punkt x fest läßt. Man kann $S(x)$ als die Spiegelung am Punkt x deuten. Beispiele für Spiegelungsräume bilden die invertierbaren Elemente einer Jordan-Algebra mit dem Produkt $x \cdot y = 2x(xy^{-1}) - x^2 y^{-1}$ und die Gruppen (und allgemeiner Moufang-Loops) mit dem Produkt $x \cdot y = xy^{-1}x$.

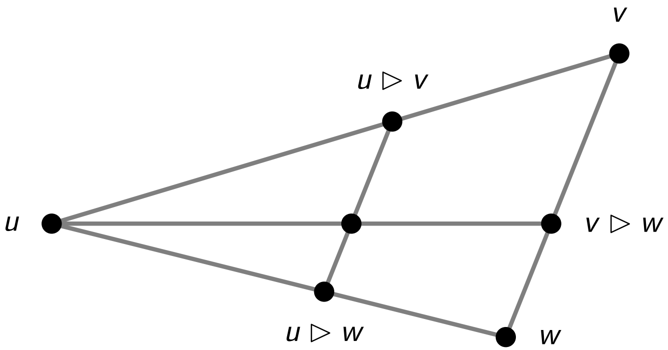
A reflection on reflections

$a \triangleright b =$ 'b reflected in a' $= 2a - b$



$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$$

From reflections to interceptions



$$v \triangleright w = (1-t)v + tw$$

$$u \triangleright (v \triangleright w) = (u \triangleright v) \triangleright (u \triangleright w) \qquad u \triangleright u = u$$

Abelian racks and quandles

Given an abelian group M with $s, t \colon M \to M$,

$$x \triangleright y = sx + ty$$

defines a rack structure if $s^2 = s(1-t)$.
These are the modules over the ring

$$\mathbb{Z}[s, t^{\pm 1}]/(s^2 = s(1-t)).$$

This rack is a quandle if $s + t = 1$, or $s = 1 - t$.
These are the modules over the ring

$$\mathbb{Z}[t^{\pm 1}].$$

Abelian group objects

C : a category with finite products

$\mathbf{Ab}(\mathbf{C})$ = category of **abelian group objects** in **C**

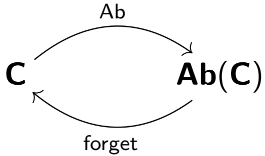
data:

$$a\colon M \times M \longrightarrow M$$
$$i\colon M \longrightarrow M$$
$$e\colon \star \longrightarrow M$$

axioms: associativity, commutativity, unit and inverse axioms

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mathrm{id}\times a} & M \times M \\ {\scriptstyle a\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle a} \\ M \times M & \xrightarrow[a]{} & M \end{array}$$

Abelianisation as an adjoint – examples



$$\begin{aligned}
\mathbf{Ab}(\mathbf{Sets}) &= \mathbf{Abel} = \mathbb{Z}\text{–}\mathbf{Mod}\\
\mathbf{Ab}(G\text{–}\mathbf{Sets}) &= \mathbb{Z}G\text{–}\mathbf{Mod}\\
\mathbf{Ab}(\mathbf{Groups}) &= \mathbf{Abel} = \mathbb{Z}\text{–}\mathbf{Mod}\\
\mathbf{Ab}(\mathbf{Quandles}) &= \mathbb{Z}[t^{\pm 1}]\text{–}\mathbf{Mod}\\
\mathbf{Ab}(\mathbf{Racks}) &= \mathbb{Z}[s, t^{\pm 1}]/(s^2 - s(1-t))\text{–}\mathbf{Mod}
\end{aligned}$$

Knots

Wir verlangen wieder, daß die Elemente S, T, U, V, W (Fig. 12) unverändert bleiben, damit die vom Ω_3-Prozeß nicht direkt betroffenen Beziehungen unberührt bleiben.

Die Gleichungen:

$$U = L_S(T),\ V' = L_S(V),\ W = L_U(V')$$

sind nach Anwendung von Ω_3 durch die Gleichungen:

$$U = L_S(T),\ V'' = L_T(V),\ W = L_S(V'')$$

zu ersetzen. Die Elimination von V' bzw: V'' führt auf

$$W = L_U L_S(V) \text{ bzw. } W = L_S L_T(V).$$

Beachten wir noch die Gleichung $U = L_S(T)$, so können wir die Invarianz gegenüber Ω_3 sichern, indem wir generell $L_{L_S(T)} L_S = L_S L_T$ fordern. Wir fassen zusammen: *Existiert zu jedem Element S einer Menge M eine bijektive Abbildung $L_S : M \to M$ mit den Eigenschaften:*

(F) $\quad L_S(S) = S,\ S \in M$

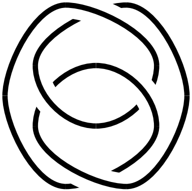
(K) $\quad L_{L_S(T)} = L_S L_T L_S^{-1},\ S, T \in M,$

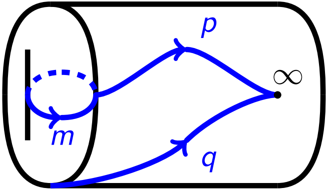
so nennen wir $\{L_S \mid S \in M\}$ eine invariante Abbildungsmenge von M.

Knots

Theorem (Joyce, Matveev, Waldhausen).
Knots K are classified by their quandles Q_K.

K





$S^3 \setminus K$

Q_K = homotopy classes of paths boundary$(S^3 \setminus K) \to \infty$

$p \triangleright q = pmp^{-1}q$ for a suitable meridian m

Primes

For a field F, let $\mathrm{Gal}(F)$ be the absolute Galois group and let

$$\mathrm{AS}(F) = \{\sigma \in \mathrm{Gal}(F) \,|\, \sigma \neq \mathrm{id} = \sigma^2\}$$

be the subspace of involutions. This is a pro-finite involutory quandle, the **Artin–Schreier quandle** of F.

Theorem (S). The Artin–Schreier quandle $\mathrm{AS}(\mathbb{Q})$ of the rational number field $\mathbb{Q}$ is a free pro-finite involutory quandle. A Cantor space of involutions gives a basis.

For $F = \mathbb{R}(\!(x)\!)(\!(y)\!)$, there are relations: $\mathrm{AS}(F) = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$.

Exercises and references