Exercises and references

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Exercises

Exercise 1. — Check that a map

$$\beta\colon X \times X \to X \times X,\ (x,y) \mapsto (x \rhd y, x)$$

induces actions of the braid groups B_n on the powers X^n of X, i.e.,

$$(\beta \times \mathrm{id})(\mathrm{id} \times \beta)(\beta \times \mathrm{id}) = (\mathrm{id} \times \beta)(\beta \times \mathrm{id})(\mathrm{id} \times \beta)$$

if and only if

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z).$$

Exercise 2. — Let $(R, \rhd)$ be a rack. Show that

$$\sigma(x) = x \rhd x$$

defines an automorphism of $(R, \rhd)$.

Exercise 3. — Show that there are, up to isomorphism, six racks with three elements.

Exercise 4. — Let (X, φ) be a permutation rack. Show that $\mathrm{Gr}(X, \varphi)$ is free abelian with basis the set X/φ of orbits.

Exercise 5. — Let $X = R$ be a rack, and $G = \mathrm{Gr}(R)$ its enveloping group. Show that the map $R \to \mathrm{Gr}(R)$ that sends an element to the corresponding left-multiplication produces a crossed G–set.

Exercise 7. — Verify that

$$(g, s) \rhd (h, t) = (gsg^{-1}h, t)$$

defines a rack structure on $\mathrm{F}(S) \times S$. What is σ for this rack?

Exercise 8. — Given an abelian group M with $s, t \colon M \to M$, show that

$$x \rhd y = sx + ty$$

defines a rack structure on M if $s^2 = s(1 - t)$. Is the converse also true? We want the addition $M \times M \to M$ to be a rack morphism.

Exercise 9. — Show that the ring $\mathbb{Z}[s, t^{\pm 1}]/(s^2 - s(1 - t))$ is a pullback of

$$\begin{array}{ccc} & & \mathbb{Z}[t^{\pm 1}] \\ & & \downarrow \\ \mathbb{Z}[t^{\pm 1}] & \longrightarrow & \mathbb{Z}, \end{array}$$

where both arrows are given by $t \mapsto 1$.

Exercise 10. — Let $\mathbb{Z}$ be the rack with $x \rhd y = 2x - y$. Is the rack $\mathbb{Z} \times \mathbb{Z}$ free?

Exercise 11. — Check the statements about the complex $\mathrm{HOM}(C, D)$:

$$(\partial f)c = \partial(fc) - (-1)^{|f|} f(\partial c)$$

is a differential. The kernel in degree 0 consists of the homomorphisms, and the image in degree 0 consists of the nullhomotopic homomorphisms $C \to D$. The zeroth homology is given by the homotopy classes of homomorphisms.

Exercise 12. — The group ring of the infinite cyclic group $\mathbb{Z}$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. Find a small free resolution of the trivial module $\mathbb{Z}$ (involving only two free modules F_0 and F_1.) Use that to compute the homology of the infinite cyclic group $\mathbb{Z}$ and give formulas for the (co)homology of all $\mathbb{Z}[t^{\pm 1}]$–modules M.

Exercise 13. — Given an abelian group A, consider the simplicial abelian group with $A_n = A$ and $\partial_j = \mathrm{id}$ for all j. Find the homology of the associated chain complex.

Exercise 14. — Given a group G, the orbit set of the orbit G/H is a singleton $\star$. What is the homotopy orbit space of G–action on G/H?

Exercise 15. — Can you describe the homology of the mapping torus of $\varphi\colon X \to X$ in terms of the homology $\mathrm{H}_\bullet(X)$ and the induced map $\mathrm{H}_\bullet(\varphi)$ on it? Test your result with $\varphi = \mathrm{id}$ and other cases of mapping tori that you can work out by other means.

Exercise 16. — Check $\partial^2 = 0$ for the differential on the rack complex. One way to do that is to write $\partial_A = \sum_j (-1)^j A_j$ and similarly $\partial_B = \sum_j (-1)^j B_j$, then you can easily check that the relations

$$\begin{aligned} \partial_A^2 &= 0 \\ \partial_B^2 &= 0 \\ \partial_A \partial_B + \partial_B \partial_A &= 0 \end{aligned}$$

hold, and these imply $\partial^2 = (\partial_A - \partial_B)^2 = 0$.

Exercise 17. — Check that a map $\varphi\colon R \times R \to A$ defines a rack structure on $R \times A$ via

$$(x, i) \rhd (y, j) = (x \rhd y, j + \varphi(x, y))$$

if and only if

$$\varphi(x \rhd y, x \rhd z) + \varphi(x, z) = \varphi(x, y \rhd z) + \varphi(y, z).$$

Exercise 18. — Check the claim about

$$H(x_1, \ldots, x_n) = (r, x_1, \ldots, x_n) :$$

it is a chain homotopy between $r\rhd$ and id:

$$(\partial H + H\partial)(x_1, \ldots, x_n) = (r \rhd x_1, \ldots, r \rhd x_n) - (x_1, \ldots, x_n)$$

Exercise 19. — In the lecture, we computed the rack homology of the trivial rack. The trivial rack is a quandle. What is its quandle homology?

Exercise 20. — Check, for instance, with a computer and RIG, that the quandle of order 6 given by the 4–cycles in S_4 has homology classes of order 4, so they are not 6–torsion.

Exercise 21. — Describe a bicomplex that has a non-trivial ∂_2 in one of its spectral sequences.

Exercise 22. Can you use the methods of the lectures to show that Quillen (co)homology for G–sets can be identified with the (co)homology of the homotopy orbit space? Or that Quillen (co)homology for groups is ordinary group (co)homology?

Disclaimer

The following is a list of references for the mini-course. It is what I have used in preparation, and it does not in any way constitute a bibliography of the subject. For reasons of time and space, I have been unable to mention many important contributors and contributions, and I am sorry for all those whom I have not been able to represent adequately.

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